OLG with MiU

Recap: Last Week

- Households maximise their lifetime utility:

$$u_t = \theta \log\left(\frac{m_{t+1}}{P_t}\right) + (1-\theta)\log\left(c_{ot+1}\right), \quad \theta \in [0,1]$$

- subject to lifetime budget constraint:

$$c_{o,t+1} = (1+r_{t+1})w_t - \left(1 + r_{t+1} - \frac{P_t}{P_{t+1}}\right)\frac{m_{t+1}}{P_t}$$

- Optimal money demand:

$$\frac{m_{t+1}}{P_t} = \theta\left(1 + \frac{1}{i_{t+1}}\right)w_t$$

- Optimal capital holding:

$$k_{t+1} = w_t - \frac{m_{t+1}}{P_t} = \left[1 - \theta\left(1 + \frac{1}{i_{t+1}}\right)\right]w_t$$

- Price level and inflation:

$$P_t = \frac{\text{money}}{\text{goods}} = \frac{Nm_t}{N(w_t - k_{t+1})} = \frac{M_t}{N(w_t - k_{t+1})} \quad \Rightarrow \quad 1+\pi_{t+1} = \frac{P_{t+1}}{P_t} = \frac{M_{t+1}}{M_t}\cdot\frac{w_t - k_{t+1}}{w_{t+1} - k_{t+2}}$$

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Production

- Production function, assuming perfect competition in labour, capital and goods markets:

$$y_t = Ak_t^{\alpha}, \quad A > 0, \alpha \in (0,1)$$

- Firms maximise profits:

$$\max_{k_t} \{Ak_t^{\alpha} - w_t - r_t k_t\} \overset{\text{FOC } (k_t)}{\Longrightarrow} A \cdot \alpha k_t^{\alpha-1} - r_t = 0 \quad \Rightarrow \quad r_t = \alpha A k_t^{\alpha-1}$$

- Perfect competition implies zero profits. So, wage:

$$w_t = Ak_t^{\alpha} - r_t k_t = Ak_t^{\alpha} - \alpha A k_t^{\alpha-1} \cdot k_t = (1-\alpha) Ak_t^{\alpha}$$

- Nominal interest rate:

$$i_{t+1} = (1 + r_{t+1})(1 + \pi_{t+1}) - 1 = (1 + \alpha A k_{t+1}^{\alpha-1}) \cdot \frac{M_{t+1}}{M_t} \cdot \frac{(1-\alpha)Ak_t^{\alpha} - k_{t+1}}{(1-\alpha)Ak_{t+1}^{\alpha} - k_{t+2}} - 1$$

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Steady State

- Capital transition equation:

$$
\begin{aligned}
k_{t+1} &= \left[1-\theta\left(1+\frac{1}{i_{t+1}}\right)\right] w_t \\
&= \left[1-\theta\left(1+\frac{1}{(1+\alpha A k_{t+1}^{\alpha-1})\cdot\frac{M_{t+1}}{M_t}\cdot\frac{(1-\alpha)Ak_t^{\alpha}-k_{t+1}}{(1-\alpha)Ak_{t+1}^{\alpha}-k_{t+2}}-1}\right)\right](1-\alpha)Ak_t^{\alpha}
\end{aligned}
$$

- Steady state ($k_t = k_{t+1} = k_{t+2} = \bar{k}$) and constant money supply ($M_t = M_{t+1} = M$):

$$
\bar{k} = \left[\frac{A(1-\theta)\alpha(1-\alpha)}{\theta+\alpha(1-\theta)}\right]^{\frac{1}{1-\alpha}}
$$

- Note: $\bar{k}$ does not depend on the money supply M.
- In the long run, $k_t \to \bar{k}$, and real variables ($y_t, c_{o,t}, r_t, w_t, P_t, i_t$) are constant, while $\pi_t = 0$.